



Dynamical Analysis of Two-Preys and One Predator Interaction Model with An Allee Effect on Predator

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Abstract

Mathematical modeling in biology is quite interesting in the field of real-world problems. This research paper focused on the interaction between two prey and one predator species. Here, our interaction is based upon the competition between two prey and one predator including an additive Allee effect in the predator population along with a Holling type II functional response. Further, this intuition allowed us to prove the positive invariance and boundedness of the model. This analysis shows that there are six equilibrium points including the coexistence of all three populations. Stability analyses are also derived and proved both locally and globally. Also in this paper, we discussed the optimal control approach to reduce the population affected by an Allee effect by the predator population. Numerical simulations are carried out to support our theoretical results.

Keywords: prey-predator model; Allee effect; equilibria analysis; stability; optimal control.

1 Introduction

One of the most fascinating subjects in theoretical ecology is the prey-predator model. Essentially, the Lotka-Volterra equation with a two-dimensional system [2] used to investigate this model. Furthermore, assuming some contests between the prey and predator may expand this to two or three dimensions. In general, the competition model Dian Savitri [19] and [18] informs us about interactions between species that share the same ecosystem. Now, when two or more species share a habitat, they share certain resources, measure their food supplies, and may even share their eating patterns. They have competition as a result of this sort of sharing. Many academics pay their way to determine the consequences of the competition (results), such as the existence and extinction of populations, as a result of such competitions [1, 9]. This work aims to investigate the prey-predator model with two prey and one predator, as well as the Allee impact on the predator.

Allee effects occur in tiny or sparse populations and are commonly considered to be widespread in nature, despite their rarity. Population growth in populations vulnerable to Allee effects slows at low densities. Warder Clyde Allee (1885-1955), a University of Chicago biologist and animal ecologist shows a specific interest in animal group behavior. Allee, a keen observer of animal behavior, discovers that crowding, not competition, inhibited population increase in many species. The Allee effect creates a loss of biodiversity for the animals in an ecosystem. Animals were dying by this affordable change in an ecosystem without finding suitable mates, affected by the disease, proper food, good environment. Many epidemiological models are formulated and published so many results [7, 8]. By influencing Allee effect parameters on the species living in an ecosystem, numerous researchers establish various concepts in both weak and strong Allee effect competition models and produced many important results [11, 24]. A positive relationship between fitness and population size can be caused by a variety of mechanisms that affect reproduction and survival. A well-established example, mate limitation, results in under-crowding in species that reproduce sexually because sexual reproduction requires contact between male and female gametes. Male limitation reduces reproduction when plants or animals release gametes into the environment or when males and females have difficulty locating each other. When behaviors such as breeding, feeding, and defense are cooperative, they become more efficient or successful in larger social groups, resulting in increased reproductive success or survivorship. Functional response, which assesses predation rate, is an essential component that influences both prey and predator growth rates.

In this study, we use the Holling type II [22] functional response. With all these credentials, this article focused on the stability and optimal control strategy for preventing the death rate of the population from an Allee impact on a predator. Raymond [14] recently develops a model with two prey and one predator, each having a Holling type II functional response, prey competition, and harvesting for all populations. Their debate focuses on the best harvesting policy for the greatest economic gain while ensuring the long-term viability of the population. Absos Ali Shaikh and Harekrishna Das [20] report the dynamics of a prey-predator model with sickness in the predator population and prey population sensitive to the Allee effect. Studying the dynamics of a prey-predator model in which both the prey and the predator exhibit herd behavior is the goal of the work of Debasis Manna, Alakes Maiti, and G.P. Samanta [12, 10]. In this study, they separate the prey population into two subpopulations: susceptible prey and infected prey, and develop a prey-predator model with Michaelis-Menten functional response [16]. S. Sourav Kumar and Joydev Chattopadhyay propose a universal prey-predator model with sickness in both the prey and the predator vulnerable to the weak Allee effects [17]. The novelty of our work is that we use the Holling-type functional response to frame the prey-predator and allow the predator population to be susceptible to the additive Allee effect. We also incorporate the idea of optimum management

to slow the rate at which species in an ecosystem are extirpating. Additionally, we create the best possible environment for the species to maintain itself in without facing extinction. We anticipate that the conclusions in this research would undoubtedly benefit ecologists and, as a result, might improve theoretical ecology. Here, we propose the model given by,

$$\begin{aligned} \frac{dR_1}{dt} &= \omega_1 R_1 \left(1 - \frac{R_1}{K_1} \right) - \lambda_{12} R_1 R_2 - \frac{\lambda_{13} R_1 D}{1 + \beta R_1}, \\ \frac{dR_2}{dt} &= \omega_2 R_2 \left(1 - \frac{R_2}{K_2} \right) - \lambda_{21} R_1 R_2 - \frac{\lambda_{23} R_2 D}{1 + \eta R_2}, \\ \frac{dD}{dt} &= -\chi D + \left(\frac{\lambda_{13} \lambda_{31} R_1 D}{1 + \beta R_1} + \frac{\lambda_{23} \lambda_{32} R_2 D}{1 + \eta R_2} \right) \left(\frac{D}{\sigma + D} \right), \end{aligned} \tag{1}$$

where, R_1, R_2 and D represent the 1st prey, 2nd prey and predator population. ω_1 and ω_2 represent the intrinsic growth rate of the 1st and 2nd prey populations. χ represents the natural death rate of the predator. K_1 and K_2 are the carrying capacity of the 1st and 2nd preys. λ_{12} and λ_{21} are the competition coefficient of 2nd prey on the 1st prey as well as 1st prey on the 2nd prey. Predation behavior is modeled by Holling type II functional response since both prey and predator are easy to capture and handle, which are given by $\frac{\lambda_{13} R_1 D}{1 + \beta R_1}$ for the 1st prey and $\frac{\lambda_{23} R_2 D}{1 + \eta R_2}$ for the 2nd prey. λ_{13} and λ_{31} are the impacts of the predator on the 1st prey and an impact of the 1st prey on a predator. Similarly, λ_{23} and λ_{32} are the impacts of the predator on the 2nd prey and an impact of the 2nd prey on a predator. And, where σ is an Allee threshold frequency.

We make the system (1) dimensionless one by choosing

$$\begin{aligned} \mu_1 &= \lambda_{12} K_2, & \mu_2 &= \lambda_{21} k_1, & Q_1 &= \lambda_{13} K_3, & Q_2 &= \lambda_{23} K_3, & \Psi_1 &= \beta K_1, \\ \Psi_2 &= \eta K_2, & e_1 &= \lambda_{31} \lambda_{13} K_1 K_3, & \text{and} & & e_2 &= \lambda_{32} \lambda_{23} K_2 K_3. \end{aligned}$$

Then, we formulate the model as,

$$\begin{aligned} \frac{du}{dt} &= \omega_1 u(1 - u) - \mu_1 uv - \frac{Q_1 uw}{1 + \Psi_1 u}, \\ \frac{dv}{dt} &= \omega_2 v(1 - v) - \mu_2 uv - \frac{Q_2 vw}{1 + \Psi_2 v}, \\ \frac{dw}{dt} &= -\chi w + \left(\frac{e_1 uw}{1 + \Psi_1 u} + \frac{e_2 vw}{1 + \Psi_2 v} \right) \left(\frac{w}{\sigma + w} \right), \end{aligned} \tag{2}$$

where u, v and w be the current population size of 1st prey, 2nd prey and predator. μ_1 and μ_2 are the competition coefficient of an interaction between the preys. $\frac{Q_1 uw}{1 + \Psi_1 u}$ and $\frac{Q_2 vw}{1 + \Psi_2 v}$ are the Holing Type II functional response. e_1 and e_2 are the rates at which predation becomes predator growth. Here, we discussed the positive invariance and the boundedness of the system (2).

1.1 Positive invariance and boundedness

Positive invariance refers to a property of a set in which any trajectory that starts inside the set remains inside the set for all time. The set is "invariant" under the dynamics of the system, according to this. It is significant since it ensures that the system will never exit a certain state space area, which can be useful for analyzing the stability characteristics of the system. For instance, all trajectories that begin inside a positive invariant set that contains a stable equilibrium point of a system will converge to that equilibrium point.

On the other hand, boundedness refers to the characteristic of a system in which all paths stay inside a certain finite area of the state space. Being confined suggests that the system will not develop into an unbounded state over time, which is frequently a bad quality. In a population model, for instance, boundedness indicates that the population size will not increase past a certain threshold, which may be set by resource limitations or other variables.

Theorem 1.1. *The solution of the system (2) remains positive across R_+^3 .*

Proof. Consider the system (2), and convert it to a vector form

$$\dot{X} = j(X(t)), \tag{3}$$

where, $X(t) = (x_1, x_2, x_3)^T = (u(t), v(t), w(t))^T$, $X(0) = (u(0), v(0), w(0))^T \in R_+^3$ and

$$j(X(t)) = \begin{pmatrix} j_1(X(t)) \\ j_2(X(t)) \\ j_3(X(t)) \end{pmatrix} = \begin{pmatrix} \omega_1 u(1-u) - \mu_1 uv - \frac{Q_1 uw}{1 + \Psi_1 u} \\ \omega_2 v(1-v) - \mu_2 uv - \frac{Q_2 vw}{1 + \Psi_2 v} \\ -\chi w + \left(\frac{e_1 uw}{1 + \Psi_1 u} + \frac{e_2 vw}{1 + \Psi_2 v} \right) \left(\frac{w}{\sigma + w} \right) \end{pmatrix}.$$

Here, $j_i(X(t)) \geq 0$, (for $i = 1, 2, 3$). Nagumos theorem [3] states that every solution of $X' = j(X(t))$ given a starting point $X(0) = X_0 \in R_+^3$ say $X(t) = X(t; X_0)$ as if $X(t) \in R_+^3 \forall t > 0$, (i.e.) (2) the solution of the system should remain positive across the territory R_+^3 . \square

Theorem 1.2. *The solutions of the system (2) are eventually bounded.*

Proof. Consider a function,

$$X = \alpha U + \beta V + W, \tag{4}$$

then the time derivative of the system (2) solutions is given by,

$$\begin{aligned} \frac{dX}{dt} &= \alpha \frac{dU}{dt} + \beta \frac{dV}{dt} + \frac{dW}{dt}, \tag{5} \\ \frac{dX}{dt} &= \alpha \left[\omega_1 u - \omega_1 u^2 - \mu_1 uv - \frac{Q_1 uw}{1 + \Psi_1 u} \right] + \beta \left[\omega_2 v - \omega_2 v^2 - \mu_2 uv - \frac{Q_2 vw}{1 + \Psi_2 v} \right] - \chi w \\ &\quad + \frac{e_1 uw}{1 + \Psi_1 u} \frac{w}{\sigma + w} + \frac{e_2 vw}{1 + \Psi_2 v} \frac{w}{\sigma + w}, \\ \frac{dX}{dt} &= \alpha \omega_1 u - \alpha \omega_1 u^2 - \alpha \mu_1 uv - \frac{\alpha Q_1 uw}{1 + \Psi_1 u} + \beta \omega_2 v - \beta \omega_2 v^2 - \beta \mu_2 uv \\ &\quad - \frac{\beta Q_2 vw}{1 + \Psi_2 v} - \chi w + \left(\frac{e_1 u}{1 + \Psi_1 u} + \frac{e_2 v}{1 + \Psi_2 v} \right) \left(\frac{w^2}{\sigma + w} \right). \end{aligned}$$

Choose $\xi > 0$,

$$\begin{aligned} \frac{dX}{dt} + \xi X &= (\alpha \omega_1 + \xi)u + (\beta \omega_2 + \xi)v + w\xi + \left(\frac{e_1 u}{1 + \Psi_1 u} + \frac{e_2 v}{1 + \Psi_2 v} \right) \frac{w^2}{\sigma + w} \\ &\quad - \alpha \omega_1 u^2 - \alpha \mu_1 uv - \frac{\alpha Q_1 uw}{1 + \Psi_1 u} - \beta \omega_2 v^2 - \beta \mu_2 uv - \frac{\beta Q_2 vw}{1 + \Psi_2 v} - \chi w, \\ \frac{dX}{dt} + \xi X &\leq (\alpha \omega_1 + \xi)u + (\beta \omega_2 + \xi)v + \xi w + \left(\frac{e_1 u}{1 + \Psi_1 u} + \frac{e_2 v}{1 + \Psi_2 v} \right) \frac{w^2}{\sigma + w}. \end{aligned}$$

As a result, a positive number exists

$$\frac{dX}{dt} + \xi X \leq M. \tag{6}$$

Hence, the suggested system’s positive solutions are eventually bounded. □

1.2 Equilibrium points

We find the equilibrium state of the model by setting $\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0$. We also suppose that the predator has a high rate of death. The probable equilibrium points for the system are listed below. $P_1(u^*, 0, 0)$, $P_2(0, v^*, 0)$, $P_3(u^*, v^*, 0)$, $P_4(u^*, 0, w^*)$, $P_5(0, v^*, w^*)$ and $P_6(u^*, v^*, w^*)$. As a result,

i) With $u^* > 0$, the equilibrium point is $P_1(u^*, 0, 0)$. In the absence of a 2nd prey and a predator, we use the system (2). Therefore,

$$P_1(u^*, 0, 0) = P_1(1, 0, 0). \tag{7}$$

ii) With $v^* > 0$, the equilibrium point is $P_2(0, v^*, 0)$. In the absence of a 1st prey and a predator ($u^* = 0$) and ($w^* = 0$). Therefore,

$$P_2(0, v^*, 0) = P_2(0, 1, 0). \tag{8}$$

iii) With both $u^* > 0$ and $v^* > 0$, therefore the equilibrium point is

$$P_3(u^*, v^*, 0) = P_3\left(\frac{\omega_2(\omega_1 - \mu_1)}{\omega_1\omega_2 - \mu_1\mu_2}, \frac{\omega_1(\omega_2 - \mu_2)}{\omega_1\omega_2 - \mu_1\mu_2}, 0\right). \tag{9}$$

This exists if $\omega_1 > \mu_1$ and $\mu_1\mu_2 < \omega_1\omega_2$. And, so $\omega_2 > \mu_2$.

iv) With $u^* > 0$ and $w^* > 0$, the equilibrium point is $P_4(u^*, 0, w^*)$. In the absence of 2nd prey ($v^* = 0$). Therefore,

$$P_4(u^*, 0, w^*) = L_1, \quad (\text{See in Appendix}).$$

v) With $v^* > 0$ and $w^* > 0$, the equilibrium point is $P_5(0, v^*, w^*)$. In the absence of 1st prey ($u^* = 0$). Therefore,

$$P_5(0, v^*, w^*) = L_2, \quad (\text{See in Appendix}).$$

The results for positive equilibrium points are produced by the cases listed below:

Necessary conditions for the equilibrium points to be positive in the absence of 2nd prey population are given by,

- For 1st prey population
 - Equilibrium Points: $P_4(u^*, 0, w^*)$

- Sign: Positive
- Case-1:

$$\begin{aligned} (\omega_1\chi\Psi_1 - \omega_1e_1) &> 0, & \omega_1e_1 < \omega_1\chi\Psi_1, \\ 4(\omega_1\chi\Psi_1 - \omega_1e_1)(-\omega_1e_1) &< (\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2, \\ \omega_1\chi\Psi_1 &> \omega_1(\chi + e_1) \text{ and } \omega_1\chi\Psi_1 > (\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2. \end{aligned}$$

- Case-2:

$$\omega_1\chi\Psi_1 < (\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2.$$

- For predator population

- Equilibrium Points: $P_4(u^*, 0, w^*)$
- Sign: Positive
- Case-1:

$$\begin{aligned} e_1Q_1 &> \chi\Psi_1Q_1, & (\omega_1\chi\Psi_1 - \omega_1e_1) > 0, & \omega_1e_1 < \omega_1\chi\Psi_1, \\ 4(\omega_1\chi\Psi_1 - \omega_1e_1)(-\omega_1\chi - \chi\sigma Q_1) &< (\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2 \text{ and} \\ \frac{\omega_1^2\chi\Psi_1e_1}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} &< \omega_1e_1 + \frac{\omega_1^2\chi e_1}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + \frac{\omega_1^2e_1^2}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + \chi\sigma\Psi_1Q_1. \end{aligned}$$

- Case-2:

$$\begin{aligned} \frac{\omega_1^2\chi\Psi_1e_1}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + (\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2 \\ < \omega_1e_1 + \frac{\omega_1^2\chi e_1}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + \frac{\omega_1^2e_1^2}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + \chi\sigma\Psi_1Q_1. \end{aligned}$$

Necessary conditions for the equilibrium points to be positive in the absence of 1st prey population are given by,

- For 2nd prey population

- Equilibrium Points: $P_5(0, v^*, w^*)$
- Sign: Positive
- Case-1:

$$\begin{aligned} (\omega_2\chi\Psi_2 - \omega_2e_2) &> 0, & \omega_2e_2 < \omega_2\chi\Psi_2, \\ 4(\omega_2\chi\Psi_2 - \omega_2e_2)(-\omega_2e_2) &< (\omega_2\chi - \omega_2\chi\Psi_2 + \omega_2e_2)^2, \\ \omega_2\chi\Psi_2 &> \omega_2(\chi + e_2) \text{ and } \omega_2\chi\Psi_2 > (\omega_2\chi - \omega_2\chi\Psi_2 + \omega_2e_2)^2. \end{aligned}$$

- Case-2:

$$\omega_2\chi\Psi_2 < (\omega_2\chi - \omega_2\chi\Psi_2 + \omega_2e_2)^2.$$

- For predator population

- Equilibrium Points: $P_5(0, v^*, w^*)$

- Sign: Positive
- Case-1:

$$\begin{aligned}
 &e_2 Q_2 > \chi \Psi_2 Q_2, \quad (\omega_2 \chi \Psi_2 - \omega_2 e_2) > 0, \quad \omega_2 e_2 < \omega_2 \chi \Psi_2, \\
 &4(\omega_2 \chi \Psi_2 - \omega_2 e_2)(-\omega_2 \chi - \chi \sigma Q_2) < (\omega_2 \chi - \omega_2 \chi \Psi_2 + \omega_2 e_2)^2, \quad \text{and} \\
 &\frac{\omega_2^2 \chi \Psi_2 e_2}{2(\omega_2 \chi \Psi_2 - \omega_2 e_2)} < \omega_2 e_2 + \frac{\omega_2^2 \chi e_2}{2(\omega_2 \chi \Psi_2 - \omega_2 e_2)} + \frac{\omega_2^2 e_2^2}{2(\omega_2 \chi \Psi_2 - \omega_2 e_2)} + \chi \sigma \Psi_2 Q_2.
 \end{aligned}$$

- Case-2:

$$\begin{aligned}
 &\frac{\omega_2^2 \chi \Psi_2 e_2}{2(\omega_2 \chi \Psi_2 - \omega_2 e_2)} + (\omega_2 \chi - \omega_2 \chi \Psi_2 + \omega_2 e_2)^2 \\
 &< \omega_2 e_2 + \frac{\omega_2^2 \chi e_2}{2(\omega_2 \chi \Psi_2 - \omega_2 e_2)} + \frac{\omega_2^2 e_2^2}{2(\omega_2 \chi \Psi_2 - \omega_2 e_2)} + \chi \sigma \Psi_2 Q_2.
 \end{aligned}$$

1.3 Existence of an endemic equilibrium

Consider the system (2). You should be comfortable with nonlinear equations by now. Indeed, you most likely spend a significant amount of time manipulating and plotting the answers to equations of the type,

$$E^* = (u, u + t_1, u + t_2), \tag{10}$$

where, t_1 and t_2 are the undefined constants. Furthermore, we wish to find the locations (u, v, w) in R^3 that satisfy the aforementioned system of non-linear equations. In this case, the polynomial equation with u^* as a root is,

$$\rho_1 u^3 + \rho_2 u^2 + \rho_3 u + \rho_4 = 0. \tag{11}$$

The co-efficients of $\rho_i, i = 1, 2, 3, 4$ are defined by,

$$\begin{aligned}
 \rho_1 &= e_1 \Psi_2 + e_2 \Psi_1 - \chi \Psi_1 \Psi_2, \\
 \rho_2 &= e_2 \Psi_1 t_2 + e_2 \Psi_1 t_1 + e_2 + e_1 \Psi_2 t_2 + e_1 \Psi_2 t_1 + e_1 - \chi \Psi_1 \Psi_2 t_2 - \chi \Psi_1 \Psi_2 t_1 - \chi \Psi_2 \\
 &\quad - \chi \Psi_1 - \chi \Psi_1 \Psi_2 \sigma, \\
 \rho_3 &= e_2 \Psi_1 t_1 t_2 + e_2 t_2 + e_2 t_1 + e_1 \Psi_2 t_1 t_2 + e_1 t_2 - \chi \Psi_1 \Psi_2 t_1 t_2 - \chi \Psi_2 t_2 - \chi \Psi_2 t_1 \\
 &\quad - \chi \Psi_1 t_2 - \chi - \chi \Psi_1 \Psi_2 \sigma t_1 - \chi \Psi_2 \sigma - \chi \Psi_1 \sigma, \\
 \rho_4 &= e_2 t_1 t_2 - \chi \Psi_2 t_1 t_2 - \chi t_2 - \chi \Psi_2 \sigma t_1 - \chi \sigma.
 \end{aligned}$$

By observing the sign of the discriminant (Descartes rule) [5], we can establish the nature of the roots for (2). As a result, there is a sufficient condition that (2) must have at least one positive root is $\rho_4 < 0$, This results in $e_2 t_1 t_2 < \chi \Psi_2 t_1 t_2 + \chi t_2 + \chi \Psi_2 \sigma t_1 + \chi \sigma$. Hence, when an endemic equilibrium and an Allee effect coexist, the system (2) can only have one positive equilibrium point.

Generally, in population dynamics stability plays a vital role to determine whether the population is stable or not at any time (t). Some researchers may develop this and talk about the stability in a continuous and discrete time too. So further, this paper is focuses to determining the stability of all three populations with an Allee effect.

2 Stability Analysis

We use the Jacobian matrix to examine the local stability of the equilibrium points,

$$J(P_i) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix},$$

where

$$u_{ij} \quad \forall (i, j = 1, 2, 3), \quad (\text{Refer Appendix}).$$

i) $P_0(0, 0, 0)$, the eigen values are obtain from the Jacobian matrix $J(P_0)$ are $\omega_1 - \lambda, \omega_2 - \lambda$ and $-\chi - \lambda$. So that, $(\omega_1 - \lambda) > 0$ and $(\omega_2 - \lambda) > 0$ are positive always. Hence, $P_0(0, 0, 0)$ is unstable.

ii) $P_1(u^*, 0, 0) = P_1(1, 0, 0)$, the eigen values are obtain from the Jacobian matrix $J(P_1)$ are $-\omega_1 - \lambda, -\omega_2 + \mu_2 + \lambda$ and $-\chi - \lambda$. So that, $-(\omega_1 + \lambda) < 0, -(\chi + \lambda) < 0$ and $\mu_2 - \omega_2 + \lambda < 0$ if $\mu_2 > \omega_2$. Hence, $P_1(u^*, 0, 0)$ is stable if $\mu_2 > \omega_2$ for $\mu_2 - \omega_2 + \lambda < 0$.

iii) $P_2(0, v^*, 0) = P_2(0, 1, 0)$, the eigen values are obtain from the Jacobian matrix $J(P_2)$ are $\omega_1 - \mu_1 - \lambda, -(\omega_2 + \lambda)$ and $-(\chi + \lambda)$. So that, $-(\omega_2 + \lambda) < 0, -(\chi + \lambda) < 0$ and $(\omega_1 - \mu_1 - \lambda) < 0$ if $\mu_1 > \omega_1$. Hence, $P_2(0, v^*, 0)$ is stable if $\mu_1 > \omega_1$ for $(\omega_1 - \mu_1 - \lambda) < 0$.

iv) $P_3(u^*, v^*, 0) = P_3\left(\frac{\omega_2(\omega_1 - \mu_1)}{\omega_1\omega_2 - \mu_1\mu_2}, \frac{\omega_1(\omega_2 - \mu_2)}{\omega_1\omega_2 - \mu_1\mu_2}, 0\right)$. The characteristic equation is used to get the eigen values of the Jacobian matrix $J(P_3)$ such as,

$$\lambda^3 - (A_1^* + B_1^* + C_1^*)\lambda^2 + (B_1^*C_1^* + A_1^*C_1^* + A_1^*B_1^* - E_1^*D_1^*)\lambda - (A_1^*B_1^*C_1^* + D_1^*C_1^*E_1^*) = 0, \quad (12)$$

where, $A_1^*, B_1^*, C_1^*, D_1^*$ and E_1^* are listed in the Appendix section.

This can be represented in the following way,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \quad (13)$$

where,

$$\begin{aligned} a_1 &= -(A_1^* + B_1^* + C_1^*), \\ a_2 &= B_1^*C_1^* + A_1^*C_1^* + A_1^*B_1^* - E_1^*D_1^*, \\ a_3 &= -(A_1^*B_1^*C_1^* + D_1^*C_1^*E_1^*). \end{aligned}$$

By, Routh-Hurwitz criteria, the negative eigenvalues are stable if $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$. Otherwise, unstable.

$$v) P_4(u^*, 0, w^*) = L_1, \quad (\text{See in Appendix}).$$

The characteristic equation is used to get the eigen values of the Jacobian matrix $J(P_4)$ such as,

$$\lambda^3 - (A_2^* + B_2^* + C_2^*)\lambda^2 + (B_2^*C_2^* + A_2^*C_2^* + E_2^*D_2^* + A_2^*B_2^*)\lambda - (A_2^*B_2^*C_2^* + D_2^*B_2^*E_2^*) = 0, \quad (14)$$

where, $A_2^*, B_2^*, C_2^*, D_2^*$ and E_2^* are listed in the Appendix section.

This can be represented in the following way,

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \quad (15)$$

where,

$$\begin{aligned} b_1 &= -(A_2^* + B_2^* + C_2^*), \\ b_2 &= B_2^*C_2^* + A_2^*C_2^* + E_2^*D_2^* + A_2^*B_2^*, \\ b_3 &= -(A_2^*B_2^*C_2^* + D_2^*B_2^*E_2^*). \end{aligned}$$

By Routh-Hurwitz criteria, the negative eigen values are stable if $b_1 > 0, b_2 > 0, b_3 > 0$ and $b_1b_2 > b_3$. Otherwise, unstable.

vi) $P_5(0, v^*, w^*) = L_2,$ (See in Appendix).

The characteristic equation is used to get the eigen values of the Jacobian matrix $J(P_5)$ such as,

$$\lambda^3 - (A_3^* + B_3^* + C_3^*)\lambda^2 + (B_3^*C_3^* + D_3^*E_3^* + A_3^*C_3^* + A_3^*B_3^*)\lambda - (A_3^*B_3^* + C_3^* + A_3^*E_3^* + D_3^*) = 0, \tag{16}$$

where, $A_3^*, B_3^*, C_3^*, D_3^*$ and E_3^* are listed in the Appendix section.

This can be represented in the following way,

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{17}$$

where,

$$\begin{aligned} C_1^* &= -(A_3^* + B_3^* + C_3^*), \\ C_2^* &= B_3^*C_3^* + D_3^*E_3^* + A_3^*C_3^* + A_3^*B_3^*, \\ C_3^* &= -(A_3^*B_3^*C_3^* + A_3^*E_3^*C_3^*). \end{aligned}$$

By Routh-Hurwitz criteria, the negative eigen values are stable if $C_1 > 0, C_2 > 0, C_3 > 0$ and $C_1C_2 > C_3$. Otherwise, unstable.

2.1 Local stability

Theorem 2.1. *The co-existence equilibrium point (u^*, v^*, w^*) of the system is locally asymptotically stable if the conditions $S_1 > 0, S_2 > 0$ and $S_1S_2 - S_3 > 0$ are satisfied, where $S_i, i = 1, 2, 3$ are given in the proof of the theorem*

Proof. The Jacobian matrix entries $J(P_i)$ at (u^*, v^*, w^*) are given [23] as follows,

$$J(P_i) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix},$$

where, $u_{ij} \forall (i, j = 1, 2, 3)$, (Refer Appendix).

The characteristic equation corresponding to $J(P_i)$ is given by

$$\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 = 0, \tag{18}$$

where, $S_i \forall (i = 1, 2, 3)$ which are listed in the Appendix section.

From the above data $S_1, S_2,$ and S_3 are greater than zero if H_1, H_2 and H_3 hold, respectively (See Appendix).

By Routh-Hurwitz criterion, if the criteria $S_1 > 0, S_2 > 0, S_3 > 0$ and $S_1S_2 - S_3 > 0$ are met the coexistence equilibrium (u^*, v^*, w^*) is locally asymptotically stable. □

2.2 Global stability

Global Stability is an important and required concept in the stability analysis of population dynamics [4, 6]. In this discussion, we consider the global stability of a system with two prey species and one predator species, considering an Allee effect. The Allee effect is a phenomenon in which the population growth rate of a species decreases at low population densities. Using the Lyapunov function, we can prove that the equilibrium point (u^*, v^*, w^*) is globally stable under certain conditions.

Theorem 2.2. *The co-existence equilibrium point $P_6(u^*, v^*, w^*)$ is globally asymptotically stable.*

Proof. Consider the Lyapunov function,

$$\begin{aligned}
 Z(u, v, w) &= (u - u^*) - u^* \log\left(\frac{u}{u^*}\right) + (v - v^*) - v^* \log\left(\frac{v}{v^*}\right) + (w - w^*) - w^* \log\left(\frac{w}{w^*}\right), \quad (19) \\
 Z(u, v, w) &= u - u^* - u^*[\log u - \log u^*] + v - v^* - v^*[\log v - \log v^*] + w \\
 &\quad - w^* - w^*[\log w - \log w^*], \\
 Z(u, v, w) &= u - u^* - u^* \log u + u^* \log u^* + v - v^* - v^* \log v + v^* \log v^* + w - w^* \\
 &\quad - w^* \log w + w^* \log w^*.
 \end{aligned}$$

Then,

$$\begin{aligned}
 \frac{\partial z}{\partial u} &= 1 - \frac{u^*}{u} = \frac{u - u^*}{u}, \\
 \frac{\partial z}{\partial v} &= 1 - \frac{v^*}{v} = \frac{v - v^*}{v}, \\
 \frac{\partial z}{\partial w} &= 1 - \frac{w^*}{w} = \frac{w - w^*}{w}.
 \end{aligned}$$

And

$$\begin{aligned}
 \frac{dz}{dt} &= \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}, \quad (20) \\
 \frac{dz}{dt} &= \frac{u - u^*}{u} \left[\omega_1 u(1 - u) - \mu_1 uv - \frac{Q_1 uw}{1 + \Psi_1 u} \right] + \frac{v - v^*}{v} \left[\omega_2 v(1 - v) - \mu_2 uv - \frac{Q_2 vw}{1 + \Psi_2 v} \right] \\
 &\quad + \frac{w - w^*}{w} \left[-\chi w + \left(\frac{e_1 uw}{1 + \Psi_1 u} + \frac{e_2 vw}{1 + \Psi_2 v} \right) \left(\frac{w}{\sigma + w} \right) \right], \\
 \frac{dz}{dt} &= (u - u^*) \left[\omega_1(1 - u) - \mu_1 v - \frac{Q_1 w}{1 + \Psi_1 u} \right] + (v - v^*) \left[\omega_2(1 - v) - \mu_2 u - \frac{Q_2 w}{1 + \Psi_2 v} \right] \\
 &\quad + (w - w^*) \left[-\chi + \left(\frac{e_1 uw}{1 + \Psi_1 u} + \frac{e_2 vw}{1 + \Psi_2 v} \right) \frac{1}{\sigma + w} \right],
 \end{aligned}$$

$$\begin{aligned} \frac{dz}{dt} &= (u - u^*) \left[\omega_1 - \omega_1 u - \mu_1 v - \frac{Q_1 w}{1 + \Psi_1 u} - \omega_1 + \omega_1 u^* + \mu_1 v^* + \frac{Q_1 w^*}{1 + \Psi_1 u^*} \right] \\ &+ (v - v^*) \left[\omega_2 - \omega_2 v - \mu_2 u - \frac{Q_2 w}{1 + \Psi_2 v} - \omega_2 + \omega_2 v^* + \mu_2 u^* + \frac{Q_2 w^*}{1 + \Psi_2 v^*} \right] \\ &+ (w - w^*) \left[-\chi + \frac{e_1 u w}{(1 + \Psi_1 u)(\sigma + w)} + \frac{e_2 v w}{(1 + \Psi_2 v)(\sigma + w)} \right. \\ &\left. + \chi - \frac{e_1 u^* w^*}{(1 + \Psi_1 u^*)(\sigma + w^*)} - \frac{e_2 v^* w^*}{(1 + \Psi_2 v^*)(\sigma + w^*)} \right], \\ \frac{dz}{dt} &= (u - u^*) \left[\omega_1 (u^* - u) + \mu_1 (v^* - v) + \frac{Q_1 w^*}{1 + \Psi_1 u^*} - \frac{Q_1 w}{1 + \Psi_1 u} \right] \\ &+ (v - v^*) \left[\omega_2 (v^* - v) + \mu_2 (u^* - u) + \frac{Q_2 w^*}{1 + \Psi_2 v^*} - \frac{Q_2 w}{1 + \Psi_2 v} \right] \\ &+ (w - w^*) \left[\frac{e_1 u w}{(1 + \Psi_1 u)(\sigma + w)} - \frac{e_1 u^* w^*}{(1 + \Psi_1 u^*)(\sigma + w^*)} + \frac{e_2 v w}{(1 + \Psi_2 v)(\sigma + w)} \right. \\ &\left. - \frac{e_2 v^* w^*}{(1 + \Psi_2 v^*)(\sigma + w^*)} \right], \\ \frac{dz}{dt} &= -(u - u^*)^2 \omega_1 - (u - u^*)(v - v^*) \mu_1 + (u - u^*) \frac{Q_1 w^*}{1 + \Psi_1 u^*} \\ &- (u - u^*) \frac{Q_1 w}{1 + \Psi_1 u} - (v - v^*)^2 \omega_2 - (v - v^*)(u - u^*) \mu_2 \\ &+ (v - v^*) \frac{Q_2 w^*}{1 + \Psi_2 v^*} - (v - v^*) \frac{Q_2 w}{1 + \Psi_2 v} + (w - w^*) \frac{e_1 u w}{(1 + \Psi_1 u)(\sigma + w)} \\ &- (w - w^*) \frac{e_1 u^* w^*}{(1 + \Psi_1 u^*)(\sigma + w^*)} + (w - w^*) \frac{e_2 v w}{(1 + \Psi_2 v)(\sigma + w)} \\ &- (w - w^*) \frac{e_2 v^* w^*}{(1 + \Psi_2 v^*)(\sigma + w^*)}. \end{aligned}$$

As a result, $\frac{dz}{dt}$ is a quadratic form that may be written as $\frac{dz}{dt} = -(X)^T A X$, where the vector is $X = (u - u^*, v - v^*, w - w^*)$ and the symmetric matrix A is given as follows,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where, $a_{11} = \omega_1, a_{12} = \mu_1, a_{21} = \mu_2, a_{22} = \omega_2$, and $a_{13} = a_{23} = a_{31} = a_{32} = a_{33} = 0$. The point $P_6(u^*, v^*, w^*)$ is globally asymptotically stable, as shown if $\frac{dz}{dt} < 0$; (i.e), the matrix A is positive definite [6]. The matrix A is now positive if $a_{11} = \omega_1 > 0, a_{12} = \mu_1 > 0, a_{21} = \mu_2 > 0, a_{22} = \omega_2 > 0$ and $a_{13} = a_{23} = a_{31} = a_{32} = a_{33} = 0, a_{11}a_{22} - a_{12}^2 > 0$ if $a_{12}^2 < a_{11}a_{22}$. This is the end of the proof. □

3 Optimal Control

An Allee effect in the predator will have an influence on the prey-predator-prey population. As a result, we may utilize some control factors to regulate the effectiveness of such catastrophes. As a result, while interacting with other species, an optimal control notion is employed to control the propagation of an Allee effect in the predator. Consider the model (2), where the population

densities of the 1st prey, 2nd prey and predator at a given time (t) are represented by $u = u(t)$, $v = v(t)$, $w = w(t)$. The system’s boundedness has previously been proven, (2), enabling us to put effective controls in place. The time-dependent controls used in this investigation will be chosen from a set of piece-wise continuous functions $[0, T]$, the set will be referred to as $PC[0, T]$. First, we consider dividing control. Make a note of the fact that you have influence over this situation (t). Our goal is to separate the first and second prey in order to create a prey-predator rivalry with an Allee effect. To do this, we let the term $\frac{Q_1x(t)uw}{1 + \Psi_1u}$ be replaced by $\frac{Q_1uw}{1 + \Psi_1u}$ in the model. And, $\frac{Q_2x^*(t)vw}{1 + \Psi_2v}$ be replaced by $\frac{Q_2vw}{1 + \Psi_2v}$.

As previously stated, the aim of using a surviving control is to minimize the rate of death caused by an Allee effect on the predator. So, instead of σ we put $(1 - y(t))\sigma$, where $y(t)$ is the rate of survival applied at time t. Clearly, the controls are limited.(i.e) $0 \leq u(t), v(t), w(t) \leq 1$ as a result, we use the admissible set.

$$X = x \in P \subset [0, T] : 0 \leq x(t) \leq \varepsilon \forall t \in [0, T]. \tag{21}$$

As a result, the controlled model is now as follows:

$$\begin{aligned} \frac{du}{dt} &= \omega_1u(1 - u) - \mu_1uw - \frac{Q_1x(t)uw}{1 + \Psi_1u}, \\ \frac{dv}{dt} &= \omega_2v(1 - v) - \mu_2uv - \frac{Q_2x^*(t)vw}{1 + \Psi_2v}, \\ \frac{dw}{dt} &= -\chi w + \left(\frac{e_1uw}{1 + \Psi_1u} + \frac{e_2vw}{1 + \Psi_2v} \right) \left(\frac{w}{(1 - y(t))\sigma + w} \right), \end{aligned} \tag{22}$$

with initial conditions,

$$\left(u(0), v(0), w(0) \right) = (u^0, v^0, w^0) \text{ and } u, v, w \in X. \tag{23}$$

Minimizing the cost function for the prey-predator competition will now meet the objectives of lowering the predator population at the end of the interval $[0, T]$ and minimizing the cost extinction over the application of the controls.

Now, first let us assume that an interaction occurs between the first prey and the predator,

$$J(x, y) := u(T) + \frac{1}{2} \int_0^T \left[A(1 - x(t))^2 + B(y(t))^2 \right] dt, \tag{24}$$

where A and B are weight parameters that are affected by the cost of implementing each control. To summarize, we have the following optimal control problems:

$$\min_{x, y \in X} J(x, y),$$

subject to (22) and (23). We will use the term denote from now on, $z = (u, v, w)$, $K_0 = (u, w)$, $i = (x, y)$, $f = (f_u, f_v, f_w)^T$ where f_u, f_v and f_w indicates the system’s right hand (22) respectively. Also, we let $\lambda = (\lambda_1 \lambda_2 \lambda_3)^T$ be the adjoint stable. From POP(Pontryagin’s Optimality Principle) [13] now, the Hamiltonian function is defined here $H(i, \lambda, z, t)$ as

$$H(i, \lambda, z, t) = \frac{1}{2} \left[A(1 - x)^2 + By^2 \right] + \lambda f,$$

$$\begin{aligned}
 H(i, \lambda, z, t) &= \frac{1}{2} \left[A(1-x)^2 + By^2 \right] + \lambda_1(t) \left[\omega_1 u(1-u) - \mu_1 uv - \frac{Q_1 xuw}{1 + \Psi_1 u} \right] \\
 &\quad + \lambda_3(t) \left[-\chi w + \frac{e_1 uw}{1 + \Psi_1 u} \frac{w}{(1-y(t))\sigma + w} \right], \\
 \frac{d\lambda_1(t)}{dt} &= \frac{\partial}{\partial u} H(t) \\
 &= -\lambda_1(t) \left[2\omega_1 u + \mu_1 v + \frac{Q_1 xw}{(1 + \Psi_1 u)^2} - w \right] - \lambda_3(t) \left[\frac{e_1 w^2}{(1 + \Psi_1 u)^2 [(y-1)\sigma + w]} \right], \tag{25} \\
 \frac{d\lambda_3(t)}{dt} &= \frac{\partial}{\partial w} H(t) \\
 &= -\lambda_1(t) \frac{Q_1 xu}{1 + \Psi_1 u} \\
 &\quad - \lambda_3(t) \left[\frac{(y-1)2e_1 uw\sigma - e_1 uw^2 - 3e_1 \Psi_1 u^2 w^2 + (y-1)2e_1 \Psi_1 u^2 w\sigma}{(\sigma - y\sigma + w + \Psi_1 u\sigma - \Psi_1 uy\sigma + \Psi_1 uw)^2} \right].
 \end{aligned}$$

Here, the transversality condition is

$$\lambda_i(T) = 0, \quad i = 1, 3, \tag{26}$$

where the transversality condition is like an upper bound for the partial derivatives of a Hamiltonian of the population. Following that, the ideal control is characterized. We get $0 < x^*(t) < \varepsilon$ we have

$$\begin{aligned}
 \frac{\partial H}{\partial x} &= -A(1-x) - \frac{\lambda_1 Q_1 uw}{1 + \Psi_1 u} = 0, \\
 x^*(t) &= \frac{\lambda_1 Q_1 uw}{A(1 + \Psi_1 u)} + 1,
 \end{aligned}$$

when $\frac{\partial H}{\partial x} < 0$,

$$-A(1-x) - \frac{\lambda_1 Q_1 uw}{1 + \Psi_1 u} < 0 \Rightarrow \frac{\lambda_1 Q_1 uw}{A(1 + \Psi_1 u)} + 1 < 0,$$

when $\frac{\partial H}{\partial x} > 0$,

$$-A(1-x) - \frac{\lambda_1 Q_1 uw}{1 + \Psi_1 u} > 0 \Rightarrow \frac{\lambda_1 Q_1 uw}{A(1 + \Psi_1 u)} + 1 > 1 \text{ but } (< \varepsilon).$$

As a result, the optimal control’s characterization is

$$x^*(t) = \min \left[\varepsilon, \max \left(\frac{\lambda_1 Q_1 uw}{A(1 + \Psi_1 u)} + 1, 0 \right) \right]. \tag{27}$$

Similarly, this is also can be derived by letting the second prey and the predator into a competition by using a Hamiltonian function. The optimal control can also be derived by using a connection with Lagrangian multipliers. The procedure continues until the consecutive iterates of control values are sufficiently close, and the updated control replaces the starting control. Hence, this proves the theorem.

4 Numerical Simulation

The system (2) is simulated using Mathematica programming in ODE solvers, and the results are shown with hypothetical parameter values to demonstrate the analytical stability properties

provided in the preceding sections. Several numerical simulations are displayed with strictly positive parameter values to gain a better understanding of the dynamics of system (2), as illustrated in the following figures. The parameter values used are not based on field data, and are simply hypothetical values used to demonstrate the dynamics of system (2).

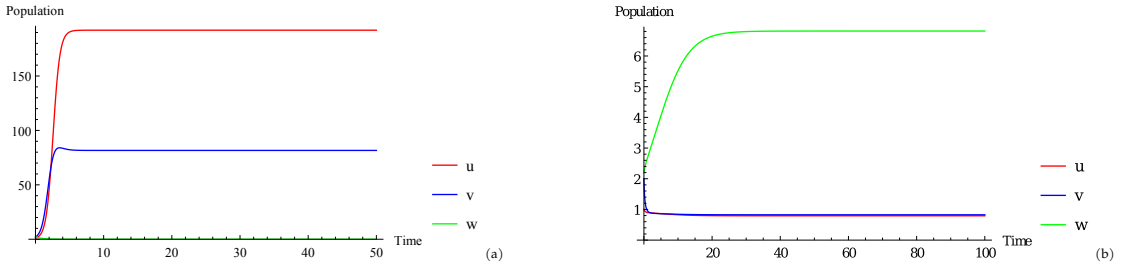


Figure 1: Existence of two prey populations with (a) $e_1 = 0.07$ and $e_2 = 0.01$. (b) $e_1 = 1.2$ and $e_2 = 0.08$.

In Figure 1: (a), the predator population vanishes when the conversion rate, at which predation becomes the predator’s growth rate, is less than 1 (i.e.) e_1 and $e_2 < 1$. In other words, if the predator population is not able to convert prey into its own population fast enough, it will die out.

In Figure 1: (b), the population of the predator species increases, leading to a decrease in the population of the other two prey populations, when the rate at which predation becomes the predator’s growth rate increases. This means that if the predator population can convert prey into its own population at a faster rate, it will grow and have a larger impact on the other populations.

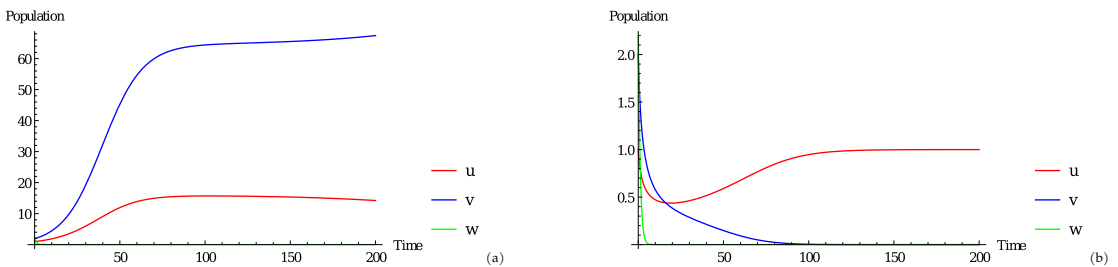


Figure 2: Existence of prey population against the time(t) can be formulated with the values, (a) $Q_1 = 0.05$ and $Q_2 = 0.03$. (b) $Q_1 = 0.02$ and $Q_2 = 0.01$.

In Figure 2: (a), we observe that when the intrinsic growth rate values are less than 1, the second prey population grows faster than the first prey population, and the predator population disappears. This suggests that the predator population cannot survive if it is not able to keep up with the growth of the prey populations.

In Figure 2: (b), we assign random values for the positive parameters to study the impact of the Holling type II functional response. We find that, when this response is used, only the first prey population continues to grow, while the predator and second prey populations go extinct. However, when the Holling type II functional response coefficient decreases, it leads to the existence of the first prey population. This implies that the predator population may be able to survive and coexist with the prey populations if it can adjust its feeding behavior based on the availability of prey.

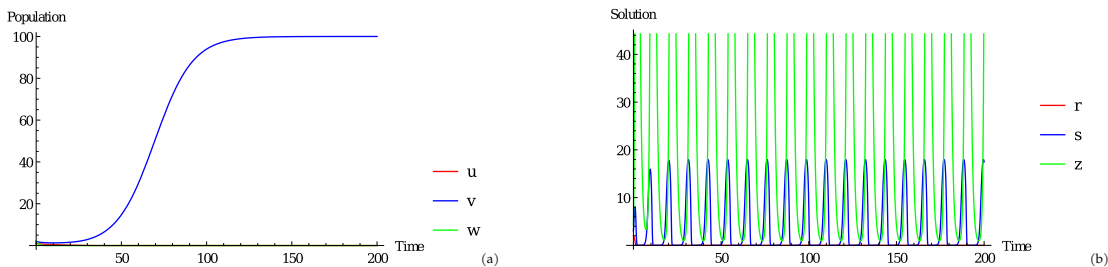


Figure 3: (a) Existence of the second prey populations when $Q_1 = 1.5, Q_2 = 1.2$ and $\sigma = 0.2$. (b) Existence of periodic solutions when an Allee threshold frequency is $\sigma = 0.01$.

In Figure 3: (a) presents a graph demonstrating that only the second prey population exists when the Holling type II functional response coefficient values are greater than 1. This suggests that the predator population is not able to consume enough of the prey to survive, and the first prey population may also be affected.

In Figure 3: (b), we observe that when an Allee effect threshold frequency value is between $0 < \sigma < 1$, the first prey population goes extinct, while the second prey and predator populations exhibit a periodic solution. This means that the populations experience cyclical changes in their numbers over time, and the presence of the Allee effect threshold frequency value may have contributed to the extinction of the first prey population. It is important to note that further research may be required to understand the specific factors that contribute to the observed population dynamics.

5 Conclusion

This paper discuss the dynamical analysis of two prey and one predator interactions along with an Allee effect on a predator. To study the dynamics of this system, a model with two prey and one predator with an Allee effect on a predator is proposed. After that we scaled the model in a dimensionless type. Then we discuss the positive invariance and boundedness of the model. The proposed model has six equilibrium points along with an endemic equilibrium point is observed in this paper. The endemic equilibrium is proved by Descartes rule. And, the stability analysis is formulated in this paper by finding the eigen values of the Jacobian matrix of the system. The stability is proven both locally and globally with the help of Routh-Hurwitz criterion and Lyapunov function. In general, disease and disaster studies require some level of control to limit the efficacy of catastrophes or the transmission of disease. Many researchers employ an optimal control strategy to limit the detrimental spread [21, 15]. So, an optimal control approach is discussed in this paper to reduce the death rate of the predator and prey which have an Allee effect and which are going to have an Allee effect in an ecosystem. Finally, numerical simulations are analyzed with the help of positive parameters of the system.

From the numerical approaches, we observe that the predator population gets extinct if the values of the conversion rate is beyond 1. And, the predator population increases when the conversion rate values are less than and greater than 1. Meanwhile, when taking intrinsic growth rate values beyond 1, that provides us to determine the second prey population grows more quickly than the first and while the predator disappears. And, the small change in numbers takes us by surprise that suddenly shows us that there is just a population of the first prey left, while the other species got extinct. Again, by changing the numbers, the second prey population can exist vice

versa is only observed in this paper. At last, when some changes occur in an Allee threshold frequency (i.e.) if $0 < \sigma < 1$ than the periodic solution occurs between the population of second prey and the predator population with respect to the time (t) while the first prey got extinct. This implies that the population of an ecosystem contains three species is bounded within the carrying capacity of an environment. Generally, the Allee threshold frequency is within $0 < \sigma < K$.

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Conflicts of Interest The authors declare that there is no conflict of interests.

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APPENDIX

$$L_1 = P_4 \left(\frac{-\omega_1\chi + \omega_1\chi\Psi_1 - \omega_1e_1 \pm \sqrt{(\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2 - 4(\omega_1\chi\Psi_1 - \omega_1e_1)(-\omega_1e_1)}}{2(\omega_1\chi\Psi_1 - \omega_1e_1)}, 0, \right. \\ \left. \frac{\omega_1e_1 + \frac{\omega_1^2\chi e_1}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} - \frac{\omega_1^2\chi\Psi_1 e_1}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + \frac{\omega_1^2 e_1^2}{2(\omega_1\chi\Psi_1 - \omega_1e_1)} + \chi\sigma\Psi_1 Q_1}{-\chi\Psi_1 Q_1 + e_1 Q_1} \right. \\ \left. \pm \frac{\omega_1e_1 \sqrt{(\omega_1\chi - \omega_1\chi\Psi_1 + \omega_1e_1)^2 - 4(\omega_1\chi\Psi_1 - \omega_1e_1)(-\omega_1\chi - \chi\sigma Q_1)2(\omega_1\chi\Psi_1 - \omega_1e_1)}}{-\chi\Psi_1 Q_1 + e_1 Q_1} \right).$$

$$L_2 = P_5 \left(0, \frac{-\omega_2\chi + \omega_2\chi\Psi_2 - \omega_2e_2 \pm \sqrt{(\omega_2\chi - \omega_2\chi\Psi_2 + \omega_2e_2)^2 - 4(\omega_2\chi\Psi_2 - \omega_2e_2)(-\omega_2e_2)}}{2(\omega_2\chi\Psi_2 - \omega_2e_2)}, \right. \\ \left. \frac{\omega_2e_2 + \frac{\omega_2^2\chi e_2}{2(\omega_2\chi\Psi_2 - \omega_2e_2)} - \frac{\omega_2^2\chi\Psi_2 e_2}{2(\omega_2\chi\Psi_2 - \omega_2e_2)} + \frac{\omega_2^2 e_2^2}{2(\omega_2\chi\Psi_2 - \omega_2e_2)} + \chi\sigma\Psi_2 e_2}{-\chi\Psi_2 Q_2 + e_2 Q_2} \right. \\ \left. \pm \frac{\omega_2e_2 \sqrt{(\omega_2\chi - \omega_2\chi\Psi_2 + \omega_2e_2)^2 - 4(\omega_2\chi\Psi_2 - \omega_2e_2)(-\omega_2\chi - \chi\sigma Q_2)2(\omega_2\chi\Psi_2 - \omega_2e_2)}}{-\chi\Psi_2 Q_2 + e_2 Q_2} \right).$$

$$u_{11} = \omega_1 - 2\omega_1u - \mu_1v - \frac{Q_1w}{(1 + \Psi_1u)^2}, \quad u_{12} = -\mu_1u, \quad u_{13} = \frac{-Q_1u}{1 + \Psi_1u}, \quad u_{21} = -\mu_2v,$$

$$u_{22} = \omega_2 - 2\omega_2v - \mu_2u - \frac{Q_2w}{(1 + \Psi_2v)^2}, \quad u_{23} = \frac{-Q_2v}{1 + \Psi_2v}, \quad u_{31} = \frac{e_1w}{(1 + \Psi_1u)^2} \frac{w}{\sigma + w},$$

$$u_{32} = \frac{e_2w}{(1 + \Psi_2v)^2} \frac{w}{\sigma + w}, \quad u_{33} = -\chi + \frac{e_1ww(2\sigma + w)}{(1 + \Psi_1u)(\sigma + w)^2} + \frac{e_2vw(2\sigma + w)}{(1 + \Psi_2v)(\sigma + w)^2},$$

$$S_1 = -(u_{11} + u_{22} + u_{33}),$$

$$S_2 = u_{11}u_{22} - \mu_1\mu_2uv + \mu_1 \frac{Q_2}{1 + \Psi_2v} uv + u_{22} \frac{Q_1}{1 + \Psi_1u} u - \frac{e_2w}{(1 + \Psi_2v)^2} \frac{w}{\sigma + w} \mu_2v \\ - u_{22} \frac{e_1w}{(1 + \Psi_1u)^2} \frac{w}{\sigma + w} + u_{22}u_{33} + \frac{Q_2v}{1 + \Psi_2v} \frac{e_2w}{(1 + \Psi_2v)^2} \frac{w}{\sigma + w},$$

$$S_3 = u_{11} \left(u_{22}u_{33} + \frac{Q_2v}{1 + \Psi_2v} \frac{e_2w}{(1 + \Psi_2v)^2} \frac{w}{\sigma + w} \right) + \mu_1u \left(-\mu_2vu_{33} + \frac{Q_2v}{1 + \Psi_2v} \frac{e_1w}{(1 + \Psi_1u)^2} \frac{w}{\sigma + w} \right) \\ - \frac{Q_1u}{1 + \Psi_1u} \left(-\mu_2v \frac{e_2w}{(1 + \Psi_2v)^2} \frac{w}{\sigma + w} - u_{22} \frac{e_1w}{(1 + \Psi_1u)^2} \frac{w}{\sigma + w} \right),$$

$$H_1 = \chi + 2\omega_1u + \mu_1v + \frac{Q_1w}{(1 + \Psi_1u)^2} + 2\omega_2v + \mu_2u + \frac{Q_2w}{(1 + \Psi_2v)^2} > (\omega_1 + \omega_2) \\ + \frac{e_1ww(2\sigma + w)}{(1 + \Psi_1u)(\sigma + w)^2} + \frac{e_1vw(2\sigma + w)}{(1 + \Psi_2v)(\sigma + w)^2},$$

$$H_2 = u_{11}u_{22} + \frac{\mu_1 Q_2}{1 + \Psi_2 v} uv + \frac{u_{22} Q_1}{1 + \Psi_1 u} u + u_{22} u_{33} + \frac{Q_2 v}{1 + \Psi_2 v} \frac{e_2 w}{(1 + \Psi_2 v)^2} \frac{w}{\sigma + w} > \mu_1 \mu_2 uv$$

$$+ \frac{e_2 w}{(1 + \Psi_2 v)^2} \frac{w}{\sigma + w} \mu_2 v + u_{22} \frac{e_1 w}{(1 + \Psi_1 u)^2} \frac{w}{\sigma + w},$$

$$H_3 = \mu_1 \mu_2 uv u_{33} < u_{11} \left(u_{22} u_{33} + \frac{Q_2 v}{1 + \Psi_2 v} \frac{e_2 w}{(1 + \Psi_2 v)^2} \frac{w}{\sigma + w} \right) + \frac{\mu_1 u Q_2 v e_1 w^2}{(1 + \Psi_2 v)(1 + \Psi_1 u)^2 (\sigma + w)}$$

$$+ \frac{Q_1 u}{1 + \Psi_1 u} \mu_2 v \frac{e_2 w}{(1 + \Psi_2 v)^2} \frac{w}{\sigma + w} + \frac{Q_1 u}{1 + \Psi_1 u} u_{22} \frac{e_1 w}{(1 + \Psi_1 u)^2} \frac{w}{\sigma + w},$$

$$A_1^* = \omega_1 - 2\omega_1 \left(\frac{\omega_2(\omega_1 - \mu_1)}{\omega_1 \omega_2 - \mu_1 \mu_2} \right) - \mu_1 \left(\frac{\omega_1(\omega_2 - \mu_2)}{\omega_1 \omega_2 - \mu_1 \mu_2} \right),$$

$$B_1^* = \omega_2 - 2\omega_2 \left(\frac{\omega_1(\omega_2 - \mu_2)}{\omega_1 \omega_2 - \mu_1 \mu_2} \right) - \mu_2 \left(\frac{\omega_2(\omega_1 - \mu_1)}{\omega_1 \omega_2 - \mu_1 \mu_2} \right),$$

$$C_1^* = -\chi, \quad D_1^* = \mu_1 \left(\frac{\omega_2(\omega_1 - \mu_1)}{\omega_1 \omega_2 - \mu_1 \mu_2} \right) \quad E_1^* = \mu_2 \left(\frac{\omega_1(\omega_2 - \mu_2)}{\omega_1 \omega_2 - \mu_1 \mu_2} \right)$$

$$A_2^* = \omega_1 - 2\omega_1 u^* - \frac{Q_1 w^*}{(1 + Psi_1 u^*)^2}, \quad B_2^* = \omega_2 - \mu_2 u^* - Q_2 w^*, \quad C_2^* = -\chi + \frac{e_1 u^* w^* (2\sigma + w)}{(1 + \Psi_1 u^*)(\sigma + w^*)^2},$$

$$D_2^* = \frac{-Q_1 u^*}{1 + \Psi_1 u^*}, \quad E_2^* = \frac{e_1 w^*}{(1 + \Psi_1 u^*)^2} \frac{w^*}{\sigma + w^*}, \quad A_3^* = \omega_1 - \mu_1 v^* - Q_1 w^*,$$

$$B_3^* = \omega_2 - 2\omega_2 v^* - \frac{Q_2 w^*}{(1 + \Psi_2 v^*)^2}, \quad C_3^* = -\chi + \frac{e_2 v^* w^* (2\sigma + w^*)}{(1 + \Psi_2 v^*)(\sigma + w^*)^2},$$

$$D_3^* = \frac{e_2 w^*}{(1 + \Psi_2 v^*)^2} \frac{w^*}{\sigma + w^*}, \quad E_3^* = \frac{-Q_2 v^*}{1 + \Psi_2 v^*}.$$